APPENDIX

MATHEMATICAL FORMULAE AND IDENTITIES

Algebra

Complete the square: if $p(x+q)^2 + r = 0$ then $x = -q \pm \sqrt{-\frac{r}{p}}$.

Modulus: $|a| = |b| \Leftrightarrow a^2 = b^2$ and $|x-a| < b \Leftrightarrow a-b < x < a+b$

Arithmetic series:

$u_n = a + (n-1)d$ $\qquad S_n = \frac{1}{2}n(a+l) = \frac{1}{2}n(a+u_n)$

Geometric series:

$u_n = ar^{n-1}$ $\qquad S_n = \frac{a(1-r^n)}{1-r} \quad (r \neq 1)$ $\qquad S_\infty = \frac{a}{1-r} \quad (|r| < 1)$

Binomial Expansion: $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$ where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Summations:

$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$ $\qquad \sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$ $\qquad \sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$

Geometry

Arc length of a circle: $s = r\theta$

Sector area of a circle: $A = \frac{1}{2}r^2\theta$

Surface area of a sphere: $A = 4\pi r^2$

Volume of a sphere: $V = \frac{4}{3}\pi r^3$

Surface area of a cone: $A = \pi r l + \pi r^2$ where l is the length of the slanted side.

Volume of a cone or pyramid: $V = \frac{1}{3}\pi r^2 h$ where h is the height of the cone.

Sine rule: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

Cosine rule: $a^2 = b^2 + c^2 - 2bc \cos A$

Trigonometric Identities

$\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$
$\sin^2\theta + \cos^2\theta \equiv 1$
$\sec^2\theta \equiv 1 + \tan^2\theta$
$\operatorname{cosec}^2\theta \equiv 1 + \cot^2\theta$

$\sin A + \sin B \equiv 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$
$\sin A - \sin B \equiv 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$
$\cos A + \cos B \equiv 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$
$\cos A - \cos B \equiv 2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

$\sin(A \pm B) \equiv \sin A\cos B \pm \cos A\sin B$
$\cos(A \pm B) \equiv \cos A\cos B \mp \sin A\sin B$
$\tan(A \pm B) \equiv \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$ where $(A \pm B) \neq \left(n + \frac{1}{2}\right)\pi$

$\sin(2A) \equiv 2\sin A\cos A$
$\cos(2A) \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$
$\tan(2A) \equiv \frac{2\tan A}{1-\tan^2 A}$ where $(2A) \neq \left(n + \frac{1}{2}\right)\pi$

$a\sin x \pm b\cos x = R\sin(x \pm \alpha)$
$a\cos x \pm b\sin x = R\cos(x \mp \alpha)$
where $R > 0$, $0 < \alpha < \frac{\pi}{2}$, $R\cos\alpha = a$, $R\sin\alpha = b$, and $R = \sqrt{a^2 + b^2}$.

Differentiation

$f(x)$	$f'(x)$
$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\arcsin x$	$\dfrac{1}{\sqrt{1-x^2}}$
$\arccos x$	$\dfrac{-1}{\sqrt{1-x^2}}$
$\arctan x$	$\dfrac{1}{1+x^2}$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\operatorname{sech}^2 x$
$\operatorname{arcsinh} x$	$\dfrac{1}{\sqrt{1+x^2}}$
$\operatorname{arccosh} x$	$\dfrac{1}{\sqrt{x^2-1}}$
$\operatorname{arctanh} x$	$\dfrac{1}{1-x^2}$

Product Rule: $\frac{d}{dx}f(x)g(x) = \left(\frac{d}{dx}f(x)\right) \times g(x) + f(x) \times \frac{d}{dx}g(x)$

Chain Rule: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

If $y = f(x)$ then $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$

Quotient Rule: $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x)-f(x)g'(x)}{\left(g(x)\right)^2}$

Integration

a is a positive constant.

$f(x)$	$\int f(x)\,dx$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\ln(\cosh x)$
$\dfrac{1}{x^2+a^2}$	$\dfrac{1}{a}\arctan\left(\dfrac{x}{a}\right)$
$\dfrac{1}{x^2-a^2}$	$\dfrac{1}{2a}\ln\left\|\dfrac{x-a}{x+a}\right\| \quad (x>a)$
$\dfrac{1}{a^2-x^2}$	$\dfrac{1}{2a}\ln\left\|\dfrac{a+x}{a-x}\right\| \quad (\|x\|<a)$
$\dfrac{1}{\sqrt{x^2+a^2}}$	$\operatorname{arcsinh}\left(\dfrac{x}{a}\right)$ $\ln\left(x+\sqrt{x^2+a^2}\right)$
$\dfrac{1}{\sqrt{x^2-a^2}}$	$\operatorname{arccosh}\left(\dfrac{x}{a}\right)$ $\ln\left(x+\sqrt{x^2-a^2}\right) \quad (x>a)$
$\dfrac{1}{\sqrt{a^2-x^2}}$	$\arcsin\left(\dfrac{x}{a}\right) \quad (\|x\|<a)$

Trapezium Rule: $\int_a^b y\,dx = \frac{1}{2}h(y_0 + 2(y_1 + y_2 + \cdots + y_{n-1}) + y_n)$ where $h = \frac{b-a}{n}$ and $y_i = f(a+ih)$

Mechanics

$$\text{Average velocity} = \frac{\text{Change in displacement}}{\text{Time taken}}$$

$$\text{Average acceleration} = \frac{\text{Change in velocity}}{\text{Time taken}}$$

Uniformly accelerated motion:

$s = ut + \frac{1}{2}at^2$

$v = u + at$

$v^2 = u^2 + 2as$

$s = \frac{1}{2}(u + v)t$

Forces:

$F = ma = \frac{d(mv)}{dt} = \frac{\text{Impulse}}{\text{Time Taken}}$

$F \leq \mu R$

$W = Fd\cos\theta$

$P = \frac{W}{t} = Fv$

Moments: $\text{Moment} = Fd\sin\theta$

Circular Motion:

$F = \frac{mv^2}{r} = mr\omega^2$

$a = \frac{v^2}{r} = r\omega^2$

Centres of mass of uniform bodies:

Triangular lamina: $\frac{2}{3}$ along median from vertex

Solid hemisphere of radius r: $\frac{3}{8}r$ from centre

Hemispherical shell of radius r: $\frac{1}{2}r$ from centre

Circular arc of radius r and angle 2α: $\frac{r\sin\alpha}{\alpha}$ from centre

Circular sector of radius r and angle 2α: $\frac{2r\sin\alpha}{3\alpha}$ from centre

Solid cone or pyramid of height h: $\frac{3}{4}h$ above the base on the line from centre of base to vertex

Conical shell of height h: $\frac{1}{3}h$ above the base on the line from centre of base to vertex

Probability and Statistics

Mean of set A and set B: $\bar{x} = \frac{n_a\overline{x_a}+n_b\overline{x_b}}{n_a+n_b}$ where n_i is the size of set i, and $\overline{x_i}$ is the mean of set i.

Mean from a frequency table: $\bar{x} = \frac{\sum xf}{\sum f}$ where f is the frequency of x.

Variance: $\sigma^2 = \frac{S_{xx}}{n}$

Standard deviation: $\sigma = \sqrt{\frac{S_{xx}}{n}}$

Summary Statistics:

$$S_{xx} = \sum(x-\bar{x})^2 = \sum x^2 - \frac{(\sum x)^2}{n}$$
$$S_{yy} = \sum y^2 - \frac{(\sum y)^2}{n}$$
$$S_{xy} = \sum xy - \frac{\sum x \sum y}{n}$$

Least squares linear regression: Line of y on x is $y = ax + b$ where $a = S_{xy}/S_{xx}$ and $b = \bar{y} - a\bar{x}$.

Product moment correlation coefficient: $r = \frac{S_{xy}}{\sqrt{S_{xx}S_{yy}}}$

Coding: If $y = \frac{x-a}{b}$ then $\bar{x} = b\bar{y} + a$ and $\sigma_x = b\sigma_y$.

Expected value of a discrete random variable: $\mathrm{E}(X) = \sum xP(X = x)$

Variance of a discrete random variable: $\mathrm{Var}(X) = \mathrm{E}\left(\left(X - \mathrm{E}(X)\right)^2\right) = \mathrm{E}(X^2) - \left(E(X)\right)^2$

Normal Distribution $N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

If Z has a normal distribution with mean 0 and variance 1, then for each value of z, the table below gives the value of $\Phi(z)$, where $\Phi(z) = P(Z \leqslant z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}t^2}\, dt$.

For negative values of z, use $\Phi(-z) = 1 - \Phi(z)$.

											ADD								
z	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359	4	8	12	16	20	24	28	32	36
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753	4	8	12	16	20	24	28	32	36
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141	4	8	12	15	19	23	27	31	35
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517	4	7	11	15	19	22	26	30	34
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879	4	7	11	14	18	22	25	29	32
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224	3	7	10	14	17	20	24	27	31
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549	3	7	10	13	16	19	23	26	29
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852	3	6	9	12	15	18	21	24	27
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133	3	5	8	11	14	16	19	22	25
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389	3	5	8	10	13	15	18	20	23
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621	2	5	7	9	12	14	16	19	21
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830	2	4	6	8	10	12	14	16	18
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015	2	4	6	7	9	11	13	15	17
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177	2	3	5	6	8	10	11	13	14
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319	1	3	4	6	7	8	10	11	13
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441	1	2	4	5	6	7	8	10	11
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545	1	2	3	4	5	6	7	8	9
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633	1	2	3	4	4	5	6	7	8
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706	1	1	2	3	4	4	5	6	6
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767	1	1	2	2	3	4	4	5	5
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817	0	1	1	2	2	3	3	4	4
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857	0	1	1	2	2	2	3	3	4
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890	0	1	1	1	2	2	2	3	3
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916	0	1	1	1	1	2	2	2	2
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936	0	0	1	1	1	1	1	2	2
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952	0	0	0	1	1	1	1	1	1
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964	0	0	0	0	1	1	1	1	1
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974	0	0	0	0	0	1	1	1	1
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981	0	0	0	0	0	0	0	1	1
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986	0	0	0	0	0	0	0	0	0

If Z has a normal distribution with mean 0 and variance 1, then, for each value of p, the table below gives the value of z such that $P(Z \leqslant z) = p$.

p	0.75	0.9	0.95	0.975	0.99	0.995	0.9975	0.999	0.9995
z	0.674	1.282	1.645	1.960	2.326	2.576	2.807	3.090	3.291