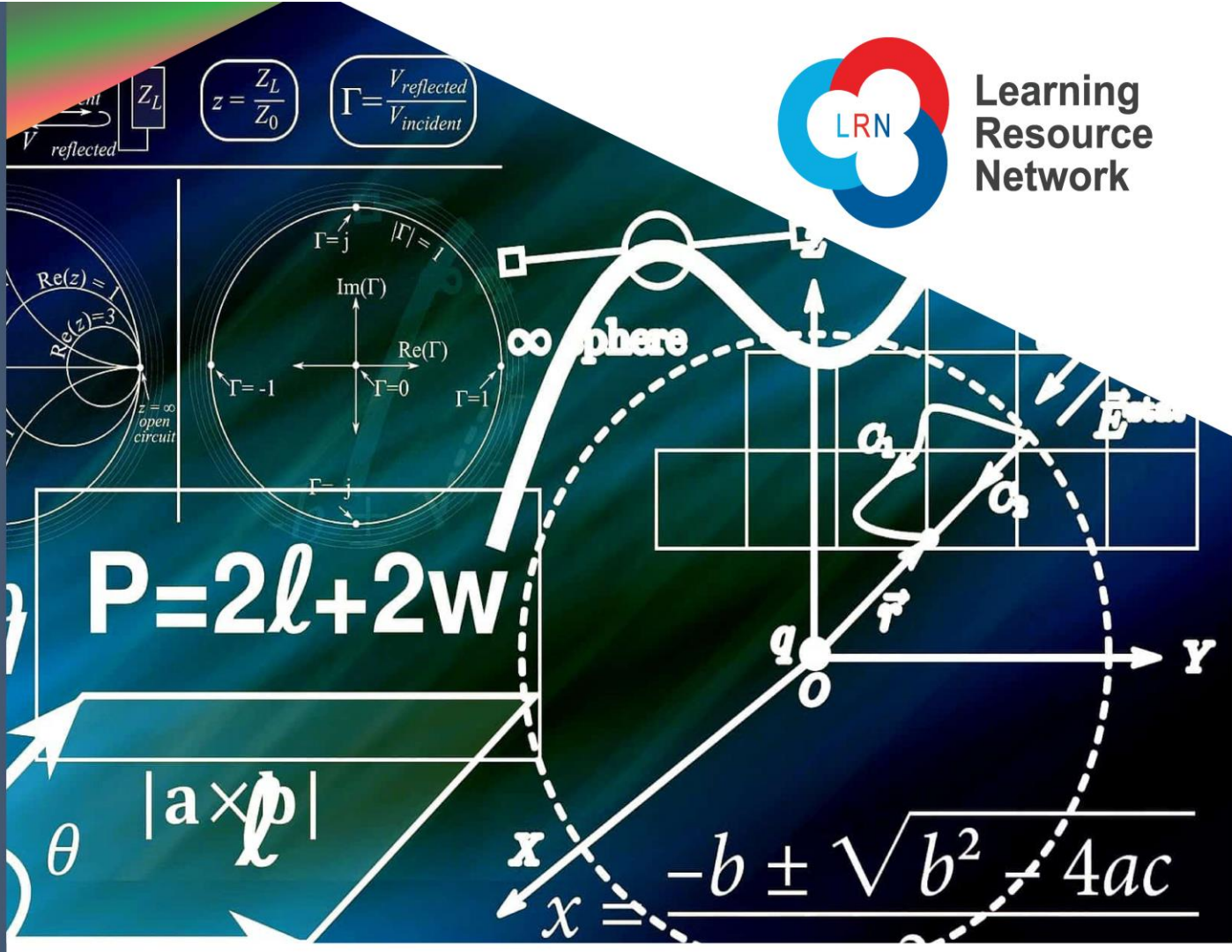
SPECIFICATION

For exams January, May and November onwards
For teaching from September 2024 onwards.



Learning Resource Network

LRN INTERNATIONAL AS AND A LEVEL

FURTHER MATHEMATICS (7140)

THE QUEEN'S AWARDS FOR ENTERPRISE: INTERNATIONAL TRADE

Contents

BACKGROUND TO LRN

Learning Resource Network (LRN) is a recognised Awarding Organisation that offers a range of qualifications to candidates, educational institutes, training providers, schools and employers.

LRN is recognised for its high quality qualifications that enable candidates to progress to other areas of study and employment in their designated fields.

In producing its qualifications, LRN uses the experience and expertise of academics, professionals working in the pertinent industries and assessment practitioners with a wealth of best practice and knowledge of validation, verification, delivery and assessment.

ACCOLADES

Queen's Award

In April 2020, LRN received the Queen's Award for Enterprise for International Trade. LRN is one of 220 organisations in the UK to be recognised with this prestigious accolade. This was in recognition of the expansion LRN brought to the overseas qualification market.

MANAGEMENT SYSTEMS

LRN has been awarded international accreditation as part of its quality controls, policies, systems and overall approach to its management systems. These awards are externally validated by the British Assessment Bureau. LRN has achieved accreditation in the form of ISO 9001: Quality Management Systems, ISO 14001: Environment Management Systems and ISO 27001: Information Security Management Systems.

CUSTOMER SERVICE EXCELLENCE

LRN has achieved the prestigious award of Customer Service Excellence. This is in recognition of its customer service practices, approach to managing and dealing with UK and Overseas customer needs, including the diverse needs of its centres.

LRN was the first UK Awarding Organisation to achieve Customer Service Excellence.
Following reaccreditation in 2019, LRN received an award for Customer Service Excellence: Compliance Plus, demonstrating that LRN went above and beyond the delivery of its customer service principles.

INTRODUCTION

This specification provides an overview to the LRN International AS & A Level Further Mathematics[1]. This document is suitable for various users, including candidates, centres, administrators, employers, parents/guardians, teachers (and other educational based staff) and examiners. The specification outlines the key features and administrative procedures required for this international qualification.

OBJECTIVE

The LRN International AS & A Level Further Mathematics is designed to enable international candidates to demonstrate their ability, in theoretical terms across a range of: Further Pure Mathematics, Further Mechanics and Further Probability and Statistics.

MODE OF DELIVERY

This qualification has been constructed to be delivered within centres. Centres will need to demonstrate to LRN, through the centre recognition processes, that they have the resources, facilities and competence to deliver. However, centres must be able to demonstrate, in line with LRN's criteria, that they have the means, capability, capacity and resources (including suitably qualified centre staff) to deliver by the method chosen by the centre.

PROGRESSION

The LRN International AS & A Level Further Mathematics has been designed to reflect the wide variation in candidates' origins, levels of education and career aims. Progression opportunities may, therefore, take a variety of paths. Depending on the level of qualification achieved, it may be appropriate for the candidate to progress to:

1. Similar level 3 qualification in Mathematics;
2. LRN Level 3 Diploma in Pre-U Foundation Studies;
3. A higher level of any qualification – e.g.; HNC/HND or Degree'
4. Vocationally Related Qualifications

[1] LRN International AS/A Level are globally recognised qualifications designed specifically for international candidates and are available outside the United Kingdom. Candidates based in England refer to the Ofqual register.

QUALIFICATION OVERVIEW

Number	Subject Content	LRN International AS Level	LRN International A Level	AO	Exam
1	Further Pure Mathematics 1	√	√	1, 2 and 3	Combination of written exam papers (externally set and marked). **AS Level** **Paper 1:** Structured Questions based on the Further Pure Mathematics (1) Duration: 2 hours **Paper 3:** Structured questions based on the Further Mechanics Duration: 1 hour 30 minutes **A Level** **Paper 2:** Structured questions Based on the Further Pure Mathematics (2) Duration: 2 hours **Paper 4:** Structured questions Based on the Further Probability and Statistics Duration: 1 hour 30 minutes
2	Further Pure Mathematics 2	–	√	1, 2 and 3	
3	Further Mechanics	√	√	1, 2 and 3	
4	Further Probability and Statistics	–	√	1, 2 and 3	

BREAKDOWN OF ASSESSMENT OBJECTIVES

AO1 - Use and apply standard techniques:

- select and correctly carry out routine procedures
- accurately recall facts, terminology and definitions

AO2 – Reason, interpret and communicate mathematically:

- construct rigorous mathematical arguments (including proofs)
- make deductions and inferences
- assess the validity of mathematical arguments
- explain their reasoning
- use mathematical language and notation correctly

AO3 – Solve problems within mathematics and in other contexts:

- translate problems in mathematical and non-mathematical contexts into mathematical processes
- interpret solutions to problems in their original context, and, where appropriate, evaluate their accuracy and limitations
- translate situations in context into mathematical models
- use mathematical models
- evaluate the outcomes of modelling in context, recognise the limitations of models and, where appropriate, explain how to refine them

ASSESSMENT

The assessment for this qualification consists of written exam papers, set and marked by the LRN.

Assessment objectives (AOs)	Weighting			
	Paper 1	Paper 2	Paper 3	Paper 4
AO1	45%	45%	40%	30%
AO2	35%	35%	35%	40%
AO3	20%	20%	25%	30%

GUIDED LEARNING HOURS (GLH)

The LRN International AS Level guided learning hours (GLH) are 180 and 360 guided learning hours for LRN International A Level. Please note the hours stated are indicative.

ENTRIES CODES

One entry per qualification is sufficient and will cover all the question papers including certification.

PRIVATE CANDIDATES

Centres are advised that private candidates are only to be enrolled with prior agreement and confirmation from LRN.

GRADING

The LRN International A Level will be graded on a six-point scale: A*, A, B, C, D and E and LRN International AS Level will be graded on a five-point scale: A, B, C, D and E Candidates who fail to reach the minimum standard for grade E will be recorded as U (unclassified) and will not receive a qualification certificate.

RESULTS

Exam series are in:

- January (results released in March)
- June (results released in August)
- November (results released in January)

RE-TAKES

Whereas candidates can re-take each paper as often as they wish, within the shelf-life of the specification.

CUSTOMER SERVICE STATEMENT

Learning Resource Network (LRN) is committed to ensuring all customers are dealt with promptly and in a professional and helpful manner. In order to guarantee this, we commit to ensuring the following in our day to day interactions with candidates, assessment centres and our stakeholder network:

- All customers will be treated equally and with respect;
- All customer information will only be used in a way which has been agreed in advance, unless we are informed of something that places them or others at risk of harm;
- All customers will be treated by staff in a professional manner.

LRN has arrangements in place to provide a telephone and e-mail helpdesk which will be staffed from 09:00 to 17:00 from Monday to Friday. Furthermore, it will respond to each e-mail, letter or telephone message it receives regarding feedback on its qualifications, centre approvals process or other matters relating to its products and/or services. The timetable for responding is as follows:

- E-mail: 5 working days
- Letter: 5 working days
- Telephone message: 5 working days

DIVERSITY AND EQUALITY

Learning Resource Network (LRN) is committed to ensuring fair and equal access to its qualifications, examinations and support materials. Our Diversity and Equality policy seeks to eliminate unjustifiable discrimination, harassment and/or victimisation and to advance equality of opportunity, thereby ensuring all candidates are treated fairly, in accordance with the protected characteristics of the Equality Act 2010. Specifically, we comply fully with the requirements laid out in the Equality Act 2010. In addition, and within the constraints of this policy, LRN will have due regard for the General data Protection Regulations (GDPR) in the retention of information which is unnecessary.

ASSESSMENT OVERVIEW AND SUMMARY

Year 1 (Outline)

a) Paper 1 – Further Pure Mathematics 1 Time: 2 hours duration

80 marks in total

There will be six-nine questions in total based on the Further Pure Mathematics 1 content
All questions should be answered in the spaces provided with working shown
The paper is externally assessed and is in the form of an written examination
This paper is compulsory for both AS and A-level Further Mathematics
The paper comprises 66% of the AS level qualification and 30% of the A-level qualification

b) Paper 3 – Further Mechanics Time: 1 hour 30 minutes duration

60 marks in total

There will be five-eight questions in total based on the Further Mechanics content
All questions should be answered in the spaces provided with working shown
The paper is externally assessed and is in the form of an written examination
This paper is compulsory for both AS and A-level Further Mathematics
The paper comprises 34% of the AS level qualification and 20% of the A-level qualification

Year 2 (Outline)

c) Paper 2 – Further Pure Mathematics 2 Time: 2 hours duration

80 marks in total

There will be six-nine questions in total based on the Further Pure Mathematics 2 content
All questions should be answered in the spaces provided with working shown
The paper is externally assessed and is in the form of an written examination
This paper is compulsory for A-level Further Mathematics
The paper comprises 30% of the A-level qualification only

d) Paper 4 – Further Probability and Statistics Time: 1 hour 30 minutes duration

60 marks in total

There will be five-eight questions in total based on the Further Probability and Statistics content
All questions should be answered in the spaces provided with working shown
The paper is externally assessed and is in the form of an written examination
This paper is compulsory for A-level mathematics
The paper comprises 20% of the A-level qualification only

1	**Further Pure Mathematics 1**		
Aim Complex further mathematics builds on the material from the A-level mathematic course. The aim of this subject content is to gain an understanding and familiarity with proofs, vectors, matrices, functions and graphs, coordinates, and polynomial equations.			
Learning Outcomes - The learner will:		**Assessment Criteria - The learner can:**	
1	Understand proof by induction	1.1	**Create** mathematical proofs utilizing the method of mathematical induction.
2	Understand vectors	2.1	**Understand** and use the vector and Cartesian forms of an equation of a straight line in 3-D.
		2.2	**Understand** and use the vector and Cartesian forms of the equation of a plane.
		2.3	**Calculate** the scalar product and use it to express the equation of a plane, and to **calculate** the angle between two lines, the angle between two planes and the angle between a line and a plane.
		2.4	**Check** whether vectors are perpendicular by using the scalar product.
		2.5	**Find** the intersection of a line and a plane.
		2.6	**Calculate** the perpendicular distance between two lines, from a point to a line and from a point to a plane.
3	Understand matrices	3.1	**Perform** operations such as addition, subtraction and multiplication with conformable matrices.
		3.2	**Multiply** a matrix by a scalar.
		3.3	**Understand** and use zero and identity matrices.
		3.4	**Use** matrices to represent linear transformations in 2-D, successive transformations and single transformations in 3-D.

		3.5	**Calculate** determinants of 2 x 2 and 3 x 3 matrices and interpret as scale factors, including the effect on orientation.
		3.6	**Understand** and use singular and non-singular matrices. Properties of inverse matrices.
		3.7	**Calculate** and use the inverse of non-singular 2 x 2 matrices and 3 x 3 matrices.
		3.8	**Solve** three linear simultaneous equations in three variables by use of the inverse matrix.
4	Understand polar coordinates	4.1	**Understand** the relations between Cartesian and polar coordinates, and convert equations of curves from Cartesian to polar form and vice versa
		4.2	**Sketch** curves with r given as a function of θ, including use of trigonometric functions.
		4.3	**Find** the area enclosed by a polar curve.
5	Understand the summation of series	5.1	**Use** the standard results for $\sum r$, $\sum r^2$, $\sum r^3$ to find related sums
		5.2	**Use** the method of differences to obtain the sum of a finite series
		5.3	**Recognise** by direct consideration of a sum to *n* terms, when a series is convergent, and find the sum to infinity in such cases.
6	Understand polynomial equations (roots)	6.1	**Remember** and apply the connections between the roots and coefficients of polynomial equations.
		6.2	**Utilize** a substitution method to derive an equation where the solutions have a straightforward connection to the solutions of the initial equation.
7	Understand rational functions and graphs	7.1	**Draw** graphs of simple rational functions, as well as determining oblique (slant) asymptotes, particularly in cases where the degree of the numerator and the denominator are at never more than 2.

		7.2	**Understand** and use relationships between the graphs of $y = f(x)$, $y^2 = f(x)$, $y = \frac{1}{f(x)}$, $y = \lvert f(x) \rvert$, $y = f(\lvert x \rvert)$

2	**Further Pure Mathematics 2**		
Aim The aims are to enable candidates to develop knowledge and understanding of further complex numbers, matrices, calculus, integration and differentiation, functions and differential equations.			
Learning Outcomes - The learner will:		**Assessment Criteria - The learner can:**	
1	Understand hyperbolic functions	1.1	**Comprehend** the meanings of the hyperbolic functions sinh x, cosh x, tanh x, sech x, cosech x, coth x using the exponential function.
		1.2	**Graph** the different hyperbolic functions.
		1.3	**Validate** and apply identities related to hyperbolic functions.
		1.4	**Know** and apply the explanations of the inverse hyperbolic functions and generate and apply the logarithmic forms.
2	Understand matrices	2.1	**Formulate** a problem involving the solution of 3 linear simultaneous equations in 3 unknowns as a problem involving the solution of a matrix equation, or vice versa
		2.2	**Understand** the cases that may arise concerning the consistency or inconsistency of 3 linear simultaneous equations, relate them to the singularity or otherwise of the corresponding matrix, solve consistent systems, and interpret geometrically in terms of lines and planes
		2.3	**Understand** the terms ‘characteristic equation’, ‘eigenvalue’ and ‘eigenvector’, as applied to square matrices
		2.4	**Find** eigenvalues and eigenvectors of 2 × 2 and 3 × 3 matrices
		2.5	**Express** a square matrix in the form QDQ^{-1}, where D is a diagonal matrix of eigenvalues and Q is a matrix whose columns are eigenvectors, and use this expression

		2.6	**Utilise** the fact that a square matrix satisfies its own characteristic equation.
3	Understand differentiation	3.1	**Differentiate** $\sin^{-1} x$, $\cos^{-1} x$, $\sinh^{-1} x$, $\cosh^{-1} x$ and $\tanh^{-1} x$ as well as hyperbolic functions
		3.2	**Calculate** to get an expression for d^2y/dx^2 in cases where the relation between the x and y variables are defined parametrically or implicitly
		3.3	**Derive** then use the first couple of terms of a Maclaurin's series for a function.
4	Understand integration	4.1	**Integrate** various hyperbolic functions and identify integrals of functions with the form $\frac{1}{\sqrt{a^2-x^2}}$, $\frac{1}{\sqrt{x^2+a^2}}$ and $\frac{1}{\sqrt{x^2-a^2}}$, and integrate functions associated with these functions. Trigonometric or hyperbolic substitutions are to be used where appropriate.
		4.2	**Derive** and then use the reduction formula in order to evaluate definite integrals
		4.3	**Know** how the area under a curve may be approximated through areas of rectangles while using rectangles to estimate or set bounds for the area under a curve. As well as to derive inequalities for limits concerning sums.
		4.4	**Utilize** integration to: • Determine arc lengths of curves described by equations in Cartesian coordinates, either with the use of a parameter or in polar coordinates. • Calculate surface areas of revolution around one of the axes for curves with equations in Cartesian coordinates, also involving the use of a parameter.
5	Understand complex numbers	5.1	**Comprehend** de Moivre's theorem, whether the exponent is a positive or negative integer, based on the geometric impact of multiplying and dividing complex numbers.
		5.2	**Demonstrate** de Moivre's theorem for a positive integer exponent.
		5.3	**Apply** de Moivre's theorem with a positive or negative rational exponent

			- to articulate trigonometric ratios of various angles as powers of the trigonometric ratios of the primary angle - to express sine and cosine powers in terms of multiple angles - in calculating series summations - in determining and utilizing the nth roots of unity.
6	Understand differential equations	6.1	**Discover** an integrating factor for a first-order linear differential equation and apply this factor to determine the general solution.
		6.2	**Remember** the definitions of 'complementary function' and 'particular integral' in the context of linear differential equations, and recall that the general solution comprises the complementary function and a particular integral.
		6.3	**Determine** the complementary solution for a linear differential equation of first or second order with constant coefficients.
		6.4	**Identify** and calculate a particular solution for a first or second order linear differential equation when a polynomial, a e^{bx}, or a cos px + b sin px function is applicable, and determine the appropriate coefficients in other cases.
		6.5	**Utilize** a specified substitution to simplify a differential equation into a linear equation of first or second order with constant coefficients, or into a first order equation with separable variables.
		6.6	**Apply** initial conditions to discover a specific solution to a differential equation, and explain the solution in relation to a problem represented by the differential equation.

3	**Further Mechanics**		
Aim The aims are to enable candidates to develop knowledge and understanding by building on their material from A-level Mechanics. Here we apply mathematical techniques to the further study of static and moving objects.			
Learning Outcomes - The learner will:		**Assessment Criteria - The learner can:**	
1	Understand projectile motion	1.1	**Demonstrate** the movement of a projectile as a particle that moves with constant acceleration and identify any limitations of this model without the use of vector methods.
		1.2	**Apply** horizontal and vertical equations of motion to solve problems related to the movement of projectiles, such as determining the velocity magnitude and direction at a specific time or position, calculating the range on a flat surface, and identifying the maximum height achieved.
		1.3	**Determine** and utilize the Cartesian equation of a projectile's trajectory, including scenarios where the initial speed and angle of projection are unknown.
2	Understand the equilibrium of a rigid body	2.1	**Determine** the moment of a force with respect to a point. .
		2.2	**Utilize** the concept that the impact of gravity on a rigid body can be represented by a single force acting at the body's center of mass, and determine the center of mass of a uniform body based on symmetry.
		2.3	**Identify** the center of mass of a triangular lamina and other simple shapes using provided information.
		2.4	**Find** the center of mass of a composite body by analyzing an equivalent system of particles in certain cases like a uniform L-shaped lamina or a uniform cone connected to a uniform hemisphere at the base.
		2.5	**Apply** the principle that a rigid body in equilibrium under coplanar forces will have a

			vector sum of zero forces and the sum of moments about any point will also be zero, and vice versa.
		2.6	**Solve** problems related to the equilibrium of a single rigid body under coplanar forces, including scenarios involving toppling or sliding
3	Understand circular motion	3.1	**Comprehend** the concept of angular speed for an object in circular motion, and utilize the equation v = $r\boldsymbol{\omega}$
		3.2	**Recognize** that the acceleration of an object moving in a circle at a consistent speed is pointed towards the center of the circle, and apply the formulas $r\omega^2$ and $\frac{v^2}{r}$.
		3.3	**Resolve** problems that can be represented by the movement of an object traveling in a horizontal circle at a steady speed
		3.4	**Address** problems that can be represented by the movement of an object in a vertical circle without any energy loss
4	Understand Hooke's law	4.1	**Utilize** Hooke's law to illustrate the relationship between the force acting on an elastic string or spring and its extension or compression, and comprehend the concept of modulus of elasticity.
		4.2	**Employ** the equation for calculating the elastic potential energy stored in a string or spring (Note: no need to demonstrate the proof of the formula).
		4.3	**Address** problems that involve forces caused by elastic strings or springs, including scenarios that require considerations of work and energy.
5	Understand linear motion (which is under a variable force)	5.1	**Address** problems that can be represented as the linear movement of a particle influenced by a changing force, through the formulation and resolution of a suitable differential equation.
		5.2	**Remember** Newton's empirical rule and the explanation of the coefficient of restitution, the characteristic $0 \leq e \leq 1$, and the definitions of the phrases 'perfectly elastic' (e = 1) and 'inelastic' (e = 0).

6	Understand momentum	6.1	**Apply** the principle of conservation of linear momentum and/or Newton's empirical law to address issues that can be represented as the direct or oblique collision of two polished spheres, or the direct or oblique collision of a polished sphere with a stationary surface.

4	**Further Probability and Statistics**		
Aim The aims are to enable candidates to develop knowledge and understanding by building on their material from A-level Probability and Statistics. Here we apply mathematical techniques to the further study of probability and statistics.			
Learning Outcomes - The learner will:		**Assessment Criteria - The learner can:**	
1	Understand (continuous) random variables	1.1	**Implement** a probability density function that can be defined in parts.
		1.2	**Apply** the general formula E (g(*X*)) = $\int f(x)g(x)dx$, where f(x) represents the probability density function of the continuous random variable X and g(X) is a function of X.
		1.3	**Comprehend** and utilize the connection between the probability density function (PDF) and the cumulative distribution function (CDF), and utilize either to calculate probabilities or percentiles.
		1.4	**Utilize** cumulative distribution functions (CDFs) of correlated variables in straightforward scenarios.
2	Understand interference (using normal and t-distributions)	2.1	**Propose** hypotheses and conduct a hypothesis test regarding the average of a population by utilizing a small sample taken from a normal population with an unknown variance, employing a t-test.
		2.2	**Compute** a combined estimation of a population variance from two different samples.
		2.3	**Create** theories regarding the contrast of population averages, and apply either a 2-sample t-test, a paired sample t-test, or a test utilizing a normal distribution, as suitable.
		2.4	**Calculate** a confidence interval for a population average, based on a small sample from a normal population with an unknown variance, using a t-distribution.

		2.5	**Calculate** a confidence interval for the distinction in population averages, utilizing a t-distribution or a normal distribution, as appropriate.
3	Understand χ2-tests	3.1	**Apply** a theoretical distribution, as specified by a hypothesis, to provided data (Questions will not require extensive computations).
		3.2	**Utilize** a χ2-test, with the correct degrees of freedom, for conducting the related goodness of fit assessment.
		3.3	**Combine** classes to ensure each expected frequency is at least 5.
		3.4	**Employ** a χ2-test, with the appropriate degrees of freedom, to test for independence in a contingency table.
4	Understand non-parametric tests	4.1	**Gain** an understanding of the concept of a non-parametric test and recognize scenarios where such a test would be beneficial. For instance, in cases where sampling from a population that cannot be assumed to have a normal distribution.
		4.2	**Comprehend** the fundamentals of the sign test, the Wilcoxon signed-rank test, and the Wilcoxon rank-sum test. This includes understanding that Wilcoxon tests are only valid for distributions that are symmetrical.
		4.3	**Apply** a single-sample sign test and a single sample Wilcoxon signed-rank test to assess a hypothesis regarding the population median.
		4.4	**Utilize** a paired-sample sign test, a Wilcoxon matched-pairs signed-rank test, and a Wilcoxon rank-sum test, as appropriate, to examine the similarity between populations.
5	Understand probability generating functions (PGF)	5.1	**Comprehend** the idea of a probability generating function (PGF) and create and apply the PGF for specified distributions.
		5.2	**Utilize** equations for the average and spread of a discrete random variable in relation to its PGF, and apply these equations to determine the average and spread

			of a provided probability distribution.
		5.3	**Apply** the concept that the PGF of the sum of independent random variables is the multiplication of the PGFs of those random variables.

APPENDIX - MATHEMATICAL FORMULAE AND IDENTITIES

FURTHER PROBABILITY & STATISTICS

Sampling and testing

Two-sample estimate of a common variance:

$$\frac{\sum(x1-\overline{x1})^2+\sum(x2-\overline{x2})^2}{n1+n2-2}$$

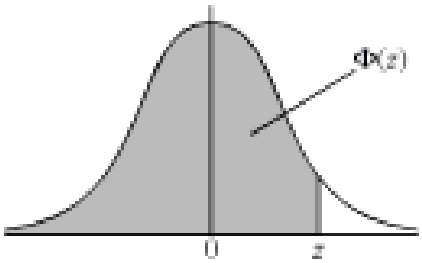
Probability generating functions

$G_x(t) = E(t^X)$, $E(X) = G'_X(1)$, $Var(X) = G''_X(1) + G'_X(1) - \{G'_X(1)\}^2$

THE NORMAL DISTRIBUTION FUNCTION

The table provides the value of Φ(z) for each value of z if Z has a normal distribution with mean 0 and variance 1.

$\Phi(z) = P(Z \leqslant z)$.



For negative values of z, use $\Phi(-z) = 1 - \Phi(z)$.

z	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
											ADD								
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359	4	8	12	16	20	24	28	32	36
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753	4	8	12	16	20	24	28	32	36
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141	4	8	12	15	19	23	27	31	35
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517	4	7	11	15	19	22	26	30	34
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879	4	7	11	14	18	22	25	29	32
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224	3	7	10	14	17	20	24	27	31
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549	3	7	10	13	16	19	23	26	29
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852	3	6	9	12	15	18	21	24	27
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133	3	5	8	11	14	16	19	22	25
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389	3	5	8	10	13	15	18	20	23
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621	2	5	7	9	12	14	16	19	21
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830	2	4	6	8	10	12	14	16	18
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015	2	4	6	7	9	11	13	15	17
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177	2	3	5	6	8	10	11	13	14
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319	1	3	4	6	7	8	10	11	13
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441	1	2	4	5	6	7	8	10	11
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545	1	2	3	4	5	6	7	8	9
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633	1	2	3	4	4	5	6	7	8
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706	1	1	2	3	4	4	5	6	6
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767	1	1	2	2	3	4	4	5	5
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817	0	1	1	2	2	3	3	4	4
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857	0	1	1	2	2	2	3	3	4
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890	0	1	1	1	2	2	2	3	3
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916	0	1	1	1	1	2	2	2	2
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936	0	0	1	1	1	1	1	2	2
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952	0	0	0	1	1	1	1	1	1
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964	0	0	0	0	1	1	1	1	1
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974	0	0	0	0	0	1	1	1	1
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981	0	0	0	0	0	0	0	1	1
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986	0	0	0	0	0	0	0	0	0

CRITICAL VALUES FOR THE NORMAL DISTRIBUTION

The table provides the value of z for each value of p if Z has a normal distribution with mean 0 and variance 1.

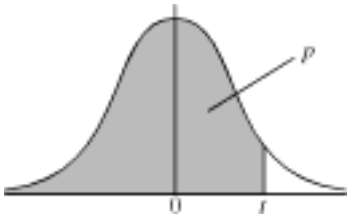
$P(Z \leqslant z) = p$.

p	0.75	0.90	0.95	0.975	0.99	0.995	0.9975	0.999	0.9995
z	0.674	1.282	1.645	1.960	2.326	2.576	2.807	3.090	3.291

CRITICAL VALUES FOR THE *t* - DISTRIBUTION

In the event that T has a t-distribution with v degrees of freedom, the table provides the value of t such that, for any combination of values of p and v:

$P(T \leqslant t) = p$.



p	0.75	0.90	0.95	0.975	0.99	0.995	0.9975	0.999	0.9995
ν = 1	1.000	3.078	6.314	12.71	31.82	63.66	127.3	318.3	636.6
2	0.816	1.886	2.920	4.303	6.965	9.925	14.09	22.33	31.60
3	0.765	1.638	2.353	3.182	4.541	5.841	7.453	10.21	12.92
4	0.741	1.533	2.132	2.776	3.747	4.604	5.598	7.173	8.610
5	0.727	1.476	2.015	2.571	3.365	4.032	4.773	5.894	6.869
6	0.718	1.440	1.943	2.447	3.143	3.707	4.317	5.208	5.959
7	0.711	1.415	1.895	2.365	2.998	3.499	4.029	4.785	5.408
8	0.706	1.397	1.860	2.306	2.896	3.355	3.833	4.501	5.041
9	0.703	1.383	1.833	2.262	2.821	3.250	3.690	4.297	4.781
10	0.700	1.372	1.812	2.228	2.764	3.169	3.581	4.144	4.587
11	0.697	1.363	1.796	2.201	2.718	3.106	3.497	4.025	4.437
12	0.695	1.356	1.782	2.179	2.681	3.055	3.428	3.930	4.318
13	0.694	1.350	1.771	2.160	2.650	3.012	3.372	3.852	4.221
14	0.692	1.345	1.761	2.145	2.624	2.977	3.326	3.787	4.140
15	0.691	1.341	1.753	2.131	2.602	2.947	3.286	3.733	4.073
16	0.690	1.337	1.746	2.120	2.583	2.921	3.252	3.686	4.015
17	0.689	1.333	1.740	2.110	2.567	2.898	3.222	3.646	3.965
18	0.688	1.330	1.734	2.101	2.552	2.878	3.197	3.610	3.922
19	0.688	1.328	1.729	2.093	2.539	2.861	3.174	3.579	3.883
20	0.687	1.325	1.725	2.086	2.528	2.845	3.153	3.552	3.850
21	0.686	1.323	1.721	2.080	2.518	2.831	3.135	3.527	3.819
22	0.686	1.321	1.717	2.074	2.508	2.819	3.119	3.505	3.792
23	0.685	1.319	1.714	2.069	2.500	2.807	3.104	3.485	3.768
24	0.685	1.318	1.711	2.064	2.492	2.797	3.091	3.467	3.745
25	0.684	1.316	1.708	2.060	2.485	2.787	3.078	3.450	3.725
26	0.684	1.315	1.706	2.056	2.479	2.779	3.067	3.435	3.707
27	0.684	1.314	1.703	2.052	2.473	2.771	3.057	3.421	3.689
28	0.683	1.313	1.701	2.048	2.467	2.763	3.047	3.408	3.674
29	0.683	1.311	1.699	2.045	2.462	2.756	3.038	3.396	3.660
30	0.683	1.310	1.697	2.042	2.457	2.750	3.030	3.385	3.646
40	0.681	1.303	1.684	2.021	2.423	2.704	2.971	3.307	3.551
60	0.679	1.296	1.671	2.000	2.390	2.660	2.915	3.232	3.460
120	0.677	1.289	1.658	1.980	2.358	2.617	2.860	3.160	3.373
∞	0.674	1.282	1.645	1.960	2.326	2.576	2.807	3.090	3.291

CRITICAL VALUES FOR THE χ^2 -DISTRIBUTION

The table provides the value of x such that P(X ⩽ x) = p for each pair of values of p and v if X has a χ^2-distribution with v degrees of freedom.

p	0.01	0.025	0.05	0.9	0.95	0.975	0.99	0.995	0.999
ν = 1	0.01571	0.09821	0.03932	2.706	3.841	5.024	6.635	7.879	10.83
2	0.02010	0.05064	0.1026	4.605	5.991	7.378	9.210	10.60	13.82
3	0.1148	0.2158	0.3518	6.251	7.815	9.348	11.34	12.84	16.27
4	0.2971	0.4844	0.7107	7.779	9.488	11.14	13.28	14.86	18.47
5	0.5543	0.8312	1.145	9.236	11.07	12.83	15.09	16.75	20.51
6	0.8721	1.237	1.635	10.64	12.59	14.45	16.81	18.55	22.46
7	1.239	1.690	2.167	12.02	14.07	16.01	18.48	20.28	24.32
8	1.647	2.180	2.733	13.36	15.51	17.53	20.09	21.95	26.12
9	2.088	2.700	3.325	14.68	16.92	19.02	21.67	23.59	27.88
10	2.558	3.247	3.940	15.99	18.31	20.48	23.21	25.19	29.59
11	3.053	3.816	4.575	17.28	19.68	21.92	24.73	26.76	31.26
12	3.571	4.404	5.226	18.55	21.03	23.34	26.22	28.30	32.91
13	4.107	5.009	5.892	19.81	22.36	24.74	27.69	29.82	34.53
14	4.660	5.629	6.571	21.06	23.68	26.12	29.14	31.32	36.12
15	5.229	6.262	7.261	22.31	25.00	27.49	30.58	32.80	37.70
16	5.812	6.908	7.962	23.54	26.30	28.85	32.00	34.27	39.25
17	6.408	7.564	8.672	24.77	27.59	30.19	33.41	35.72	40.79
18	7.015	8.231	9.390	25.99	28.87	31.53	34.81	37.16	42.31
19	7.633	8.907	10.12	27.20	30.14	32.85	36.19	38.58	43.82
20	8.260	9.591	10.85	28.41	31.41	34.17	37.57	40.00	45.31
21	8.897	10.28	11.59	29.62	32.67	35.48	38.93	41.40	46.80
22	9.542	10.98	12.34	30.81	33.92	36.78	40.29	42.80	48.27
23	10.20	11.69	13.09	32.01	35.17	38.08	41.64	44.18	49.73
24	10.86	12.40	13.85	33.20	36.42	39.36	42.98	45.56	51.18
25	11.52	13.12	14.61	34.38	37.65	40.65	44.31	46.93	52.62
30	14.95	16.79	18.49	40.26	43.77	46.98	50.89	53.67	59.70
40	22.16	24.43	26.51	51.81	55.76	59.34	63.69	66.77	73.40
50	29.71	32.36	34.76	63.17	67.50	71.42	76.15	79.49	86.66
60	37.48	40.48	43.19	74.40	79.08	83.30	88.38	91.95	99.61
70	45.44	48.76	51.74	85.53	90.53	95.02	100.4	104.2	112.3
80	53.54	57.15	60.39	96.58	101.9	106.6	112.3	116.3	124.8
90	61.75	65.65	69.13	107.6	113.1	118.1	124.1	128.3	137.2
100	70.06	74.22	77.93	118.5	124.3	129.6	135.8	140.2	149.4

WILCOXON SIGNED-RANK TEST

The size of the sample is n.

The ranks that correspond to the positive differences add up to P. The ranks that correspond to the negative disparities are added together to form Q. Of P and Q, T is the smaller.

The table provides the highest value of T for each number of n, which, at the specified level of significance, will result in the null hypothesis being rejected.

Critical values of *T*

	Level of significance			
One-tailed	0.05	0.025	0.01	0.005
Two-tailed	0.1	0.05	0.02	0.01
$n = 6$	2	0		
7	3	2	0	
8	5	3	1	0
9	8	5	3	1
10	10	8	5	3
11	13	10	7	5
12	17	13	9	7
13	21	17	12	9
14	25	21	15	12
15	30	25	19	15
16	35	29	23	19
17	41	34	27	23
18	47	40	32	27
19	53	46	37	32
20	60	52	43	37

WILCOXON RANK-SUM TEST

The sizes of the two samples are m and n, where m ⩾ n.

The total of the item rankings in the sample of size m is denoted by Rm. Rm and m(n + m + 1) – Rm are both smaller than W.
The table provides the highest value of W for each pair of m and n values, which will result in the null hypothesis being rejected at the specified level of significance.

Critical values of *W*

	Level of significance											
One-tailed	0.05	0.025	0.01	0.05	0.025	0.01	0.05	0.025	0.01	0.05	0.025	0.01
Two-tailed	0.1	0.05	0.02	0.1	0.05	0.02	0.1	0.05	0.02	0.1	0.05	0.02
n	*m* = 3			*m* = 4			*m* = 5			*m* = 6		
3	6	–	–									
4	6	–	–	11	10	–						
5	7	6	–	12	11	10	19	17	16			
6	8	7	–	13	12	11	20	18	17	28	26	24
7	8	7	6	14	13	11	21	20	18	29	27	25
8	9	8	6	15	14	12	23	21	19	31	29	27
9	10	8	7	16	14	13	24	22	20	33	31	28
10	10	9	7	17	15	13	26	23	21	35	32	29

	Level of significance											
One-tailed	0.05	0.025	0.01	0.05	0.025	0.01	0.05	0.025	0.01	0.05	0.025	0.01
Two-tailed	0.1	0.05	0.02	0.1	0.05	0.02	0.1	0.05	0.02	0.1	0.05	0.02
n	*m* = 7			*m* = 8			*m* = 9			*m* = 10		
7	39	36	34									
8	41	38	35	51	49	45						
9	43	40	37	54	51	47	66	62	59			
10	45	42	39	56	53	49	69	65	61	82	78	74

Summations

$$\sum_{r=1}^{n} r = \tfrac{1}{2}n(n+1), \qquad \sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1), \qquad \sum_{r=1}^{n} r^3 = \tfrac{1}{4}n^2(n+1)^2$$

Maclaurin's Series

$$\mathrm{f}(x) = \mathrm{f}(0) + x\mathrm{f}'(0) + \frac{x^2}{2!}\mathrm{f}''(0) + \ldots + \frac{x^r}{r!}\mathrm{f}^{(r)}(0) + \ldots$$

$$\mathrm{e}^x = \exp(x) = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^r}{r!} + \ldots \qquad \text{(all } x\text{)}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{r+1}\frac{x^r}{r} + \ldots \qquad (-1 < x \leqslant 1)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^r \frac{x^{2r+1}}{(2r+1)!} + \ldots \qquad \text{(all } x\text{)}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^r \frac{x^{2r}}{(2r)!} + \ldots \qquad \text{(all } x\text{)}$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots + (-1)^r \frac{x^{2r+1}}{2r+1} + \ldots \qquad (-1 \leqslant x \leqslant 1)$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + \frac{x^{2r+1}}{(2r+1)!} + \ldots \qquad \text{(all } x\text{)}$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + \frac{x^{2r}}{(2r)!} + \ldots \qquad \text{(all } x\text{)}$$

$$\tanh^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots + \frac{x^{2r+1}}{2r+1} + \ldots \qquad (-1 < x < 1)$$

Hyperbolic Functions

$$\cosh^2 x - \sinh^2 x \equiv 1, \qquad \sinh 2x \equiv 2\sinh x \cosh x, \qquad \cosh 2x \equiv \cosh^2 x + \sinh^2 x$$

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$$

$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right) \qquad (x \geqslant 1)$$

$$\tanh^{-1} x = \tfrac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \qquad (|x| < 1)$$

Trigonometry

If $t = \tan\frac{1}{2}x$ then:

$$\sin x = \frac{2t}{1+t^2} \qquad \text{and} \qquad \cos x = \frac{1-t^2}{1+t^2}$$

Differentiation

$f(x)$	$f'(x)$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\operatorname{sech}^2 x$
$\sinh^{-1} x$	$\frac{1}{\sqrt{1+x^2}}$
$\cosh^{-1} x$	$\frac{1}{\sqrt{x^2-1}}$
$\tanh^{-1} x$	$\frac{1}{1-x^2}$

Integration (Note: Omitted are arbitrary constants. α shows a positive constant).

$f(x)$	$\int f(x)\,dx$	
$\sec x$	$\ln\lvert\sec x+\tan x\rvert=\ln\lvert\tan(\frac{1}{2}x+\frac{1}{4}\pi)\rvert$	$(\lvert x\rvert<\frac{1}{2}\pi)$
$\operatorname{cosec} x$	$-\ln\lvert\operatorname{cosec} x+\cot x\rvert=\ln\lvert\tan(\frac{1}{2}x)\rvert$	$(0<x<\pi)$
$\sinh x$	$\cosh x$	
$\cosh x$	$\sinh x$	
$\operatorname{sech}^2 x$	$\tanh x$	
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\frac{x}{a}\right)$	$(\lvert x\rvert<a)$
$\frac{1}{\sqrt{x^2-a^2}}$	$\cosh^{-1}\left(\frac{x}{a}\right)$	$(x>a)$
$\frac{1}{\sqrt{a^2+x^2}}$	$\sinh^{-1}\left(\frac{x}{a}\right)$	

Mechanics

Uniformly accelerated motion

$v = u + at$, $s = \frac{1}{2}(u+v)t$, $s = ut + \frac{1}{2}at^2$, $v^2 = u^2 + 2as$

Further Mechanics

Motion of a projectile

Equation of trajectory is:

$$y = x\tan\theta - \frac{gx^2}{2V^2\cos^2\theta}$$

Elastic strings and springs:

$$T = \frac{\lambda x}{l}, \qquad E = \frac{\lambda x^2}{2l}$$

Motion in a circle:

$\omega^2 r$ or $\frac{v^2}{r}$

Centres of mass of uniform bodies:

Triangular lamina: $\frac{2}{3}$ along median from vertex

Solid hemisphere of radius r: $\frac{3}{8}r$ from centre

Hemispherical shell of radius r: $\frac{1}{2}r$ from centre

Circular arc of radius r and angle 2α: $\frac{r\sin\alpha}{\alpha}$ from centre

Circular sector of radius r and angle 2α: $\frac{2r\sin\alpha}{3\alpha}$ from centre

Solid cone or pyramid of height h: $\frac{3}{4}h$ from vertex